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Manifest spin 2 duality with electric and magnetic sources

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ABSTRACT. We extend the formulation of spin 2 fields on Minkowski space which makes the action manifestly invariant under duality rotations to the case of interactions with external electric and magnetic sources by adding suitable potentials for the longitudinal and trace parts. In this framework, the string singularity of the linearized Taub-NUT solution is resolved into a Coulomb-like solution. Suitable surface charges to measure energy-momentum and angular momentum of both electric and magnetic type are constructed.

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Contents

1 Introduction

Duality rotations for massless spin 2 fields have recently been shown to be symmetries of the action in the context of a Hamiltonian formulation involving suitable potentials that arise when solving the constraints [1]. This symmetry can be extended to higher spin fields [2] and also to the case of spin 2 fields propagating on an (A)dS background [3, 4], but does not survive gravitational self-interactions [5].

The spin 2 result generalizes the so-called double potential formalism for spin 1 fields [6, 7], which has been extended so as to include couplings to dynamical dyons by using Dirac strings [8, 9]. In the original double potential formalism, Gauss's constraint is solved in terms of new transverse vector potentials for the electric field so that electromagnetism is effectively formulated on a reduced phase space with all gauge invariance eliminated. Alternatively, one may choose [10] to double the gauge redundancy of standard electromagnetism by using a description with independent vector and longitudinal potentials for the magnetic and electric fields and 2 scalar potentials that appear as Lagrange multipliers for the electric and magnetic Gauss constraints. In this framework, the string-singularity of the solution describing a static dyon is resolved into a Coulomb-like solution. Furthermore, magnetic charge no longer appears as a topological conservation law but as a surface charge on a par with electric charge.

The aim of the present work is to apply the same strategy to the spin 2 case. Doubling the gauge invariance by keeping all degrees of freedom of symmetric tensors now leads to a second copy of linearized lapse and shifts as Lagrange multipliers for the new magnetic constraints. As a consequence, the string singularity of the gravitational dyon, the linearized Taub-NUT solution is resolved and becomes Coulomb-like exactly as the purely electric linearized Schwarzschild solution. Furthermore, as required by manifest duality, magnetic mass, momentum and Lorentz charges also appear as surface integrals.

Our work thus presents a manifestly duality invariant alternative to [11] where the coupling of spin 2 fields to conserved electric and magnetic sources has been achieved in a manifestly Poincaré invariant way through the introduction of Dirac strings.

Recent and not so recent related work includes for instance [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24] and references therein.

In section **2**, we briefly recall the two ingredients needed for our formulation: the Hamiltonian description of spin 2 fields propagating on Minkowski spacetime and the decomposition of symmetric tensors into their irreducible components, giving rise to the reduced phase space description of linearized gravity.

Our analysis starts in section **3** with a degree of freedom count that shows that the phase space of duality invariant spin 2 fields with doubled gauge invariance can be taken to consist of 2 symmetric tensors, 2 vectors and 2 scalars in 3 dimensions. We then

define the metric, extrinsic curvature and their duals in terms of the phase space variables and propose our duality invariant action principle with enhanced gauge invariance. We proceed by identifying the canonically conjugate pairs and discuss the gauge structure, Hamiltonian and duality generators of the theory. In the absence of sources, we then show how the generators for global Poincaré transformations of Pauli-Fierz theory, reviewed in detail in appendix **A**, can be extended to the duality invariant theory.

The coupling to external electric and magnetic sources is discussed in section **4**. The equations of motion are first solved in the simplest case of a point-particle dyon sitting at the origin. They are Coulomb-like without string singularities. By identifying the Riemann tensor in terms of the canonical variables and computing it for this case, we show in appendix **B** that this solution indeed describes the linearized Taub-NUT solution.

In section **5** we discuss the surface charges of the theory and show that they include electric and magnetic energy-momentum and angular momentum. Because of the non-locality of the Poisson structure, we proceed indirectly and show that the expressions obtained by generalizing the surface charges of Pauli-Fierz theory in a duality invariant way fulfill the standard properties. Finally, we investigate how the surface charges transform under a global Poincaré transformation of the sources.

2 Preliminaries

2.1 Canonical formulation of Pauli-Fierz theory

The Hamiltonian formulation of general relativity linearized around flat spacetime is

$$S_{PF}[h_{mn},\pi^{mn},n_m,n]=\int dt\Big[\int d^3x\,\big(\pi^{mn}\dot h_{mn}-n^m\mathcal{H}_m-n\mathcal{H}\big)-H_{PF}\Big], \tag{2.1}$$

with

$$H_{PF}[h_{mn},\pi^{mn}]=\int d^3x\big(\pi^{mn}\pi_{mn}-\frac{1}{2}\pi^2+\frac{1}{4}\partial^r h^{mn}\partial_r h_{mn}- \\ -\frac{1}{2}\partial_m h^{mn}\partial^r h_{rn}+\frac{1}{2}\partial^m h\partial^n h_{mn}-\frac{1}{4}\partial^m h\partial_m h\big), \tag{2.2}$$

and

$$\mathcal{H}_m=-2\partial^n\pi_{mn},\quad \mathcal{H}_\perp=\Delta h-\partial^m\partial^n h_{mn}. \tag{2.3}$$

Here, indices are lowered and raised with the flat space metric δ_{mn} and its inverse, $h = h^m{}_m$, $\pi=\pi^m{}_m$ and $\Delta=\partial_m\partial^m$ is the Laplacian in flat space. The linearized 4 metric is reconstructed using $h_{00}=-2n$ and $h_{0i}=n_i$.

2.2 Decomposition of symmetric rank two tensors

Symmetric rank two tensors ϕ_{mn} decompose as [25, 26]

$$\phi_{mn} = \phi^{TT}_{mn} + \phi^{T}_{mn} + \phi^{L}_{mn}, \tag{2.4}$$
$$\phi^{L}_{mn} = \partial_m \psi_n + \partial_n \psi_m, \tag{2.5}$$
$$\phi^{T}_{mn} = \frac{1}{2}\left(\delta_{mn}\Delta - \partial_m\partial_n\right)\psi^T. \tag{2.6}$$

Here ϕ^{TT}_{mn} is the transverse-traceless part, containing two independent components. The tensor ϕ^{T}_{mn} contains the trace of the transverse part of ϕ_{mn} and only one independent component. The last three components are the longitudinal part contained in ϕ^{L}_{mn}. In terms of the original tensor ϕ_{mn} the potentials for the longitudinal part and the trace are given by

$$\psi_m = \Delta^{-1}\left(\partial^n\phi_{mn} - \frac{1}{2}\Delta^{-1}\partial_m\partial^k\partial^l\phi_{kl}\right), \tag{2.7}$$
$$\psi^T = \Delta^{-1}\left(\phi - \Delta^{-1}\partial^m\partial^n\phi_{mn}\right), \tag{2.8}$$

while the transverse-traceless part is then defined as the remainder,

$$\phi^{TT}_{mn} = \phi_{mn} - \phi^{T}_{mn} - \phi^{L}_{mn}. \tag{2.9}$$

This implies

$$\begin{aligned}\Delta^2\phi^{TT}_{mn} &= \Delta^2\phi_{mn} - \Delta\partial_m\partial^k\phi_{kn} - \Delta\partial_n\partial^k\phi_{km} \\ &\quad - \frac{1}{2}\Delta(\delta_{mn}\Delta - \partial_m\partial_n)\phi + \frac{1}{2}(\delta_{mn}\Delta + \partial_m\partial_n)\partial^k\partial^l\phi_{kl},\end{aligned} \tag{2.10}$$
$$\begin{aligned}\int d^3x\, \phi^{mn}\Delta^2\phi^{TT}_{mn} &= \int d^3x\left(\Delta\phi^{mn}\Delta\phi_{mn} + 2\partial_m\phi^{mn}\Delta\partial^k\phi_{kn} - \frac{1}{2}(\Delta\phi)^2\right. \\ &\quad \left. + \partial_m\partial_n\phi^{mn}\Delta\phi + \frac{1}{2}\partial_m\partial_n\phi^{mn}\partial_k\partial_l\phi^{kl}\right).\end{aligned} \tag{2.11}$$

Alternatively, one can introduce the local operator $\mathcal{P}^{TT}$

$$\left(\mathcal{P}^{TT}\phi\right)_{mn} = \frac{1}{2}\left[\epsilon_{mpq}\partial^p(\Delta\phi^q{}_n - \partial_n\partial_r\phi^{qr}) + \epsilon_{npq}\partial^p(\Delta\phi^q{}_m - \partial_m\partial_r\phi^{qr})\right], \tag{2.12}$$

which projects out the longitudinal and trace parts and onto a transverse-traceless tensor,

$$\left(\mathcal{P}^{TT}\phi\right)_{mn} = \mathcal{P}^{TT}\left(\phi^{TT}\right)_{mn} = \left(\mathcal{P}^{TT}\phi\right)^{TT}_{mn}. \tag{2.13}$$

In addition,

$$\left(\mathcal{P}^{TT}(\mathcal{P}^{TT}\phi)\right)_{mn} = -\Delta^3\phi^{TT}_{mn}. \tag{2.14}$$

As a consequence, the transverse-traceless tensor ϕ^{TT}_{mn} can be written as $\mathcal{P}^{TT}$ acting on a suitable potential ψ^{TT}_{mn},

$$\phi^{TT}_{mn} = \left(\mathcal{P}^{TT}\psi^{TT}\right)_{mn}, \quad \psi^{TT}_{mn} = -\Delta^{-3}\left(\mathcal{P}^{TT}\phi\right)_{mn}. \tag{2.15}$$

The operator $\mathcal{P}^{TT}$ is related to the way the Hamiltonian constraint $\mathcal{H} = 0$ is solved by expressing the metric h_{mn} in terms of superpotentials in [1]. When acting on a transverse-traceless tensor, the two terms of (2.12) involving $\partial_r\phi^{qr}$ can be dropped. In this case, $\mathcal{P}^{TT}$ is related to the generalized curl [2, 5],

$$(\mathcal{O}\phi)_{mn} = \frac{1}{2}(\epsilon_{mpq}\partial^p\phi^q{}_n + \epsilon_{npq}\partial^p\phi^q{}_m), \tag{2.16}$$

$$\left(\mathcal{P}^{TT}\phi^{TT}\right)_{mn} = \Delta\left(\mathcal{O}\phi^{TT}\right)_{mn}. \tag{2.17}$$

A second operator that projects out the longitudinal and trace parts and onto a transverse-traceless tensor is $\mathcal{Q}^{TT}$,

$$\left(\mathcal{Q}^{TT}\phi\right)_{mn} = \epsilon_{mpq}\epsilon_{nrs}\partial^p\partial^r\Delta\phi^{qs} - \frac{1}{2}(\delta_{mn}\Delta - \partial_m\partial_n)(\Delta\phi - \partial^p\partial^r\phi_{pr}). \tag{2.18}$$

In this case,

$$\left(\mathcal{Q}^{TT}(\mathcal{Q}^{TT}\phi)\right)_{mn} = \Delta^4\phi^{TT}_{mn}, \tag{2.19}$$

so that the transverse-traceless tensor ϕ^{TT}_{mn} can be written as $\mathcal{Q}^{TT}$ acting on another potential χ^{TT}_{mn},

$$\phi^{TT}_{mn} = \left(\mathcal{Q}^{TT}\chi^{TT}\right)_{mn}, \quad \chi^{TT}_{mn} = \Delta^{-4}\left(\mathcal{Q}^{TT}\phi\right)_{mn}. \tag{2.20}$$

In turn this operator is related to the way the constraints $\mathcal{H}_m = 0$ are solved by expressing the momenta π^{mn} in terms of superpotentials in [1]. When acting on a transverse-traceless tensor, the last term can again be dropped and it is related to the square of the generalized curl,

$$\left(\mathcal{Q}^{TT}\phi^{TT}\right)_{mn} = \Delta\left(\mathcal{O}(\mathcal{O}\phi^{TT})\right)_{mn} = -\Delta^2\phi^{TT}_{mn}. \tag{2.21}$$

The elements of the decomposition are orthogonal under integration if boundary terms can be neglected,

$$\int d^3x\, \phi^{mn}\varphi_{mn} = \int d^3x\left(\phi^{TTmn}\varphi^{TT}_{mn} + \phi^{Lmn}\varphi^{L}_{mn} + \phi^{Tmn}\varphi^{T}_{mn}\right). \tag{2.22}$$

and the operators $\mathcal{P}^{TT}, \mathcal{Q}^{TT}, \mathcal{O}$ are self-ajoint, e.g.,

$$\int d^3x\, \left(\mathcal{P}^{TT}\phi\right)^{mn}\varphi_{mn} = \int d^3x\, \phi^{mn}\left(\mathcal{P}^{TT}\varphi\right)_{mn}. \tag{2.23}$$

2.3 Reduced phase space for linearized gravity

For completeness, let us briefly recall [25] the reduced phase space associated with the Pauli-Fierz action. Because of the orthogonality of the decomposition, the canonically conjugate pairs can be directly read off from the kinetic term and are given by

$$\big(h^{TT}_{mn}(x),\, \pi^{kl}_{TT}(\vec{y})\big), \quad \big(h^{L}_{mn}(\vec{x}),\, \pi^{kl}_{L}(y)\big), \quad \big(h^{T}_{mn}(x),\, \pi^{kl}_{T}(y)\big). \tag{2.24}$$

The first class constraints $\mathcal{H}_m = 0 = \mathcal{H}$ are equivalent to $\pi^{kl}_L = 0 = h^T_{mn}$. They can be gauge fixed through the conditions $h^L_{mn} = 0 = \pi^{kl}_T$. The reduced theory only depends on 2 degrees of freedom (per spacetime point), the transverse-traceless components $(h^{TT}_{mn}(\vec{x}),\, \pi^{kl}_{TT}(y))$ and the reduced Hamiltonian simplifies to

$$H^R = \int d^3x \left(\pi^{mn}_{TT}\pi^{TT}_{mn} + \frac{1}{4}\partial_r h^{TT}_{mn}\partial^r h^{mn}_{TT}\right). \tag{2.25}$$

3 Action and symmetries

3.1 Degree of freedom count

In order to be able to couple to sources of both electric and magnetic type in a duality invariant way, we want to keep all components and double the gauge invariance of the theory. With 2 degrees of freedom, $\#\,\mathrm{dof} = 2$, and 8 first class constraints, $\#\,\mathrm{fcc} = 8$, we thus need 10 canonical pairs, $\#\,\mathrm{cp} = 10$, according to the degree of freedom count [27]

$$2 * (\#\,\mathrm{cp}) = 2 * (\#\,\mathrm{dof}) + 2 * (\#\,\mathrm{fcc}). \tag{3.1}$$

This can be done by taking 2 symmetric tensors, 2 vectors and 2 scalars as fundamental canonical variables,

$$z^A = (H^a_{mn}, A^a_m, C^a). \tag{3.2}$$

3.2 Change of variables and duality rotations

For $a = 1, 2$, consider $h^a_{mn} = (h_{mn}, h^D_{mn})$ and $\pi^{mn}_a = (\pi^{mn}_D, \pi^{mn})$ and the definitions

$$\begin{aligned}
h^a_{mn} &= \epsilon_{mpq}\partial^p H^{aq}{}_n + \epsilon_{npq}\partial^p H^{aq}{}_m + \partial_m A^a_n + \partial_n A^a_m + \frac{1}{2}(\delta_{mn}\Delta - \partial_m\partial_n)C^a \\
&= 2\Delta^{-1}\left(\mathcal{P}^{TT} H^a\right)_{mn} + \partial_m\big(\Delta^{-1}\epsilon_{npq}\partial^p\partial_r H^{aqr} + A^a_n\big) \\
&\qquad + \partial_n\big(\Delta^{-1}\epsilon_{mpq}\partial^p\partial_r H^{aqr} + A^a_m\big) + \frac{1}{2}(\delta_{mn}\Delta - \partial_m\partial_n)C^a,
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\pi^a_{mn} &= \epsilon_{mpq}\epsilon_{nrs}\partial^p\partial^r H^{aqs} - \partial_m\partial^r H^a_{rn} - \partial_n\partial^r H^a_{rm} - (\delta_{mn}\Delta - \partial_m\partial_n)H^a + \delta_{mn}\partial^k\partial^l H^a_{kl} \\
&= \Delta^{-1}\left(\mathcal{Q}^{TT} H^a\right)_{mn} - \partial_m\partial^r H^a_{rn} - \partial_n\partial^r H^a_{rm} - \frac{1}{2}(\delta_{mn}\Delta - \partial_m\partial_n)H^a \\
&\qquad + \frac{1}{2}\Delta^{-1}(\delta_{mn}\Delta + \partial_m\partial_n)\partial^p\partial^r H^a_{pr} \\
&= -\Delta H^a_{mn}.
\end{aligned} \tag{3.4}$$

The relations for $h_{mn}[H^1, A^1, C^1]$ and $\pi^{mn}[H^2]$ are the local change of coordinates from the standard canonical variables of linearized gravity to the new variables. They are invertible and, as usual, the inverse is not local. The relations for $h^2_{mn} = h^D_{mn}$, $\pi^{mn}_1 = \pi^{mn}_D$ serve to denote convenient combinations of the new variables in terms of which expressions below will simplify. As indicated by the notation, the infinitesimal duality rotations among the fundamental variables are

$$\delta_D H^a_{mn} = \epsilon^{ab} H_{bmn},\ \delta_D A^a_m = \epsilon^{ab} A_{bm},\ \delta_D C^a = \epsilon^{ab} C_b. \tag{3.5}$$

Here, ϵ_{ab} is skew-symmetric with $\epsilon_{12} = 1$ and indices are lowered and raised with δ_{ab} and its inverse. Duality invariance will be manifest if all the internal indices a are contracted with the invariant tensors $\delta_{ab}, \epsilon_{ab}$. Since h^a_{mn}, π^{mn}_a are linear combinations of the fundamental variables, they are rotated in exactly the same way. We can thus consider $h^2_{mn} = h^D_{mn}$, $\pi^{mn}_1 = \pi^{mn}_D$ as the dual spatial metric and the dual extrinsic curvature in the linearized theory.

3.3 Action principle and locality

The duality invariant local action principle that we propose is of the form

$$S_G[z^A, u^\alpha] = \int d^4x\, (a_A[z]\dot{z}^A - u^\alpha\gamma_\alpha[z]) - \int dt\, H[z], \tag{3.6}$$

where u^α denote the 8 Lagrange multipliers and γ_α the constraints.

Let us stress here that we use the assumption that the flat space Laplacian Δ is invertible in order to show equivalence with the usual Hamiltonian or covariant formulation of Pauli-Fierz theory and also to disentangle the canonical structure. The action principle (3.6) itself and the associated equations of motion will be local both in space and

in time independently of this assumption. The theory itself is not local as a Hamiltonian gauge theory (see e.g. [28], chapter 12) because the Poisson brackets among the canonical variables will not be local.

3.4 Canonical structure

The explicit expression for the kinetic term is

$$a_A \dot{z}^A = \epsilon_{ab} H^{amn}\Big(\big(\mathcal{P}^{TT} \dot{H}^b\big)_{mn} + \partial_m \Delta \dot{A}^b_n + \partial_n \Delta \dot{A}^b_m + \\ + \frac{1}{2}(\delta^{mn}\Delta - \partial^m\partial^n)\Delta \dot{C}^b\Big). \quad (3.7)$$

The canonically conjugate pairs are identified by writing the integrated kinetic term as

$$\int d^4x\, a_A \dot{z}^A = \int d^4x \Big(-2\Delta \left(\mathcal{O} H^2_{TT}\right)^{mn} \dot{H}^{1TT}_{mn} + 2\Delta \partial_m H^{2mn}_L \dot{A}^1_n \\ -2\Delta\partial_m \dot{H}^{1mn}_L A^2_n - \frac{1}{2}\Delta(\Delta H^2_T - \partial_p\partial_q H^{2pq}_T)\dot{C}^1 + \frac{1}{2}\Delta(\Delta H^1_T - \partial_p\partial_q H^{1pq}_T)\dot{C}^2\Big). \quad (3.8)$$

This means that the usual canonical pairs of linearized gravity can be choosen in terms of the new variables as

$$\Big(H^{1TT}_{mn}(x),\, -2\Delta \left(\mathcal{O} H^2_{TT}\right)^{kl}(y)\Big),\, \Big(C^1(x),\, -\frac{1}{2}\Delta(\Delta H^2_T - \partial_p\partial_q H^{2pq}_T)\,(y)\Big), \\ \Big(A^1_m(x),\, 2\Delta\partial_r H^{2rn}_L(y)\Big), \quad (3.9)$$

The 4 additional canonical pairs are

$$\Big(A^2_m(x),\, -2\Delta\partial_r H^{1rn}_L(y)\Big), \Big(C^2(x),\, \frac{1}{2}\Delta(\Delta H^1_T - \partial_p\partial_q H^{1pq}_T)(y)\Big). \quad (3.10)$$

In particular, it follows that

$$\{h^a_{mn}(x), \pi^{bkl}(y)\} = \epsilon^{ab}\frac{1}{2}(\delta^k_m\delta^l_n + \delta^k_m\delta^l_n)\delta^{(3)}(x,y). \quad (3.11)$$

3.5 Gauge structure

The constraints $\gamma_\alpha \equiv (\mathcal{H}_{am}, \mathcal{H}_{a\perp})$ are choosen as

$$\mathcal{H}_{am} = -2\epsilon_{ab}\partial^n \pi^b_{mn} = 2\epsilon_{ab}\Delta\partial^n H^b_{mn}, \quad (3.12)$$
$$\mathcal{H}_{a\perp} = \Delta h_a - \partial_m\partial_n h^{mn}_a = \Delta^2 C_a. \quad (3.13)$$

They are first class and abelian

$$\{\gamma_\alpha, \gamma_\beta\} = 0. \quad (3.14)$$

The constraints $\mathcal{H}_{1m}, \mathcal{H}_{1\perp}$ are those of the standard Hamiltonian formulation of Pauli-Fierz theory expressed in terms of the new variables. The new constraints $\gamma^N_\Delta = 0$ are

$$\mathcal{H}_{2m} = 0 = \mathcal{H}_{2\perp}. \tag{3.15}$$

They are equivalent to $\partial^r H^1_{rm} = 0 = C^2$ and are gauge fixed through the conditions $A^2_m = 0 = H^{1T}_{mn}$. This does not affect π^{2kl}, while h^1_{mn} is changed by a gauge transformation. The partially gauge fixed theory corresponds to the usual Pauli-Fierz theory in Hamiltonian form as described in section **2.1**.

More precisely, the observables of a Hamiltonian field theory with constraints are defined as equivalence classes of functionals that have weakly vanishing Dirac brackets with the constraints and where two functionals are considered as equivalent if they agree on the surface defined by the constraints (see e.g. [28]). The new constraints together with the gauge fixing conditions form second class constraints. The Dirac bracket algebra of observables of this (partially) gauge fixed formulation is isomorphic to the Poisson bracket algebra of observables of the extended formulation on the one hand, and to the Poisson bracket algebra of observables of Pauli-Fierz theory on the other hand.

In the same way, the original constraints $\mathcal{H}_{1m} = 0 = \mathcal{H}_{1\perp}$ are equivalent to $\partial^r H^2_{rm} = 0 = C^1$ and are gauge fixed through $A^2_m = 0 = H^{2T}_{mn}$, leading to the completely reduced theory in terms of the 2 transverse-traceless physical degrees of freedom.

If $\varepsilon^\alpha = (\xi^{am}, \xi^{a\perp})$ collectively denote the gauge parameters, the gauge symmetries are canonically generated by the smeared constraints,

$$\delta_\varepsilon z^A = \{z^A, \Gamma[\varepsilon]\}, \quad \Gamma[\varepsilon] = \int d^3x\, \gamma_\alpha \epsilon^\alpha, \tag{3.16}$$

so that

$$\delta_\varepsilon H^a_{mn} = -\Delta^{-1}\epsilon^{ab}(\delta_{mn}\Delta - \partial_m\partial_n)\xi^\perp_b, \quad \delta_\varepsilon A^a_m = \xi^a_m, \quad \delta_\xi C^a = 0, \tag{3.17}$$

which implies in particular

$$\delta_\varepsilon h^a_{mn} = \partial_m \xi^a_n + \partial_n \xi^a_m, \qquad \delta_\varepsilon \pi^a_{mn} = \epsilon^{ab}(\delta^{mn}\Delta - \partial^m\partial^n)\xi^\perp_b. \tag{3.18}$$

Note that a way to get local gauge transformations for the fundamental variables is to multiply the constraints by Δ, which is allowed when the flat space Laplacian is invertible. This amounts to introducing suitable potentials for the gauge parameters and Lagrange multipliers.

3.6 Duality generator

The canonical generator for the infinitesimal duality rotations (3.5) is

$$\begin{aligned} D = \int d^3x \Big(&- (\mathcal{P}^{TT} H^a)_{mn} H_a^{mn} + 2\Delta \partial_r H_a^{rm} A_m^a \\ &- \frac{1}{2}\Delta(\Delta H^a - \partial^m \partial^n H_{mn}^a) C_a \Big) \\ \approx - \int d^3x\, &\mathcal{P}^{TT}(H^a)_{mn} H_a^{mn}. \end{aligned} \tag{3.19}$$

The duality generator is only weakly gauge invariant,

$$\{\mathcal{H}_{am}, D\} = \epsilon_{ab}\mathcal{H}_m^b \qquad \{\mathcal{H}_{a\perp}, D\} = \epsilon_{ab}\mathcal{H}_\perp^b. \tag{3.20}$$

On the constraint surface, it coincides with the generator found in [1] up to normalisation, where it has been cast in the form of a Chern-Simons term.

3.7 Hamiltonian

In terms of the new variables (3.3)-(3.4), the Pauli-Fierz Hamiltonian reads

$$\begin{aligned} H_{PF} = \int d^3x \Big(&H^{amn}\Delta^2 H_{amn}^{TT} - 2\Delta\partial^r H_{rn}^2 \partial_s H^{2sn} - \\ &- \partial^r\partial^s H_{rs}^2 \Delta H^2 - \frac{1}{2}(\partial^r \partial^s H_{rs}^2)^2 + \frac{1}{8}\Delta C^1 \Delta^2 C^1 \Big), \end{aligned} \tag{3.21}$$

where one can use (2.10) to expand the first term as a local functional of H_{mn}^a.

The local Hamiltonian $H = \int d^3x\, h$ of the manifestly duality invariant action principle (3.6) is

$$\begin{aligned} H = \int d^3x \Big(&H^{amn}\Delta^2 H_{amn}^{TT} - 2\Delta\partial^r H_{rn}^a \partial_s H_a^{sn} - \\ &- \partial^r\partial^s H_{rs}^a \Delta H_a - \frac{1}{2}\partial^r\partial^s H_{rs}^a \partial_k \partial_l H_a^{kl} + \frac{1}{8}\Delta C^a \Delta^2 C_a \Big) \\ = \int d^3x \Big(&\Delta H_{mn}^a \Delta H_a^{mn} - \frac{1}{2}\Delta H^a \Delta H_a + \frac{1}{8}\Delta C^a \Delta^2 C_a \Big). \end{aligned} \tag{3.22}$$

It is equivalent to the Pauli-Fierz Hamiltonian since it reduces to the latter when the additional constraints $\partial^r H_{rm}^1 = 0 = C^2$ hold. Note that the terms proportional to $\partial^r H_{rm}^a$ and C^a may be dropped since they vanish on the constraint surface, $H \approx \int d^3x\, H^{amn}\Delta^2 H_{amn}^{TT}$.

The Hamiltonian is gauge invariant on the constraint surface,

$$\{H, \Gamma[\xi]\} = \int d^3x\, \mathcal{H}_m^a \partial^m \xi_a^\perp. \tag{3.23}$$

In order for the action (3.6) to be gauge invariant, it follows from (3.23) that the Lagrange multipliers u^α need to transform as

$$\delta_\xi u^{am} = \dot{\xi}^{am} - \partial^m \xi^{a\perp}, \qquad \delta_\xi u^{a\perp} = \dot{\xi}^{a\perp}. \tag{3.24}$$

3.8 Poincaré generators

Consider now a symmetry generator of Pauli-Fierz theory. It is defined by an observable $K[h^1, \pi^2]$ whose representative is weakly conserved in time,

$$\frac{\partial}{\partial t} K + \{K, H_{PF}\} \approx 0. \tag{3.25}$$

Since the new Hamiltonian differs from the Pauli-Fierz one by terms proportional to the new constraints $\gamma^N_\Delta = 0$ given explicitly in (3.15), we have

$$H = H_{PF} + \int d^3x\, \gamma^N_\Delta k^\Delta. \tag{3.26}$$

Furthermore, since K, when expressed in terms of the new variables, does not depend on H^2_L, H^1_T, so that $\{K, \int d^3x\, \gamma^N_\Delta k^\Delta\} \approx 0$ in the extended theory, it follows that K is also weakly conserved and thus a symmetry generator of the extended theory,

$$\frac{\partial}{\partial t} K + \{K, H\} \approx 0. \tag{3.27}$$

Consider then the Poincaré generators $Q_G(\omega, a)$ of Pauli-Fierz theory as described in Appendix **A**. When expressed in terms of the new variables, they are representatives for the Poincaré generators of the extended theory. Indeed, we just have shown that they are symmetry generators, while we have argued in Section **3.5** that their Poisson algebra is isomorphic when restricted to their respective constraint surfaces.

Since symmetry generators form a Lie algebra with respect to the Poisson bracket, $\{Q_G(\omega, a), D\}$ is a symmetry generator. In much the same way as for the Hamiltonian, we now want to show that one can find representatives for the Poincaré generators that are duality invariant,

$$\{Q^D_G(\omega, a), D\} = 0, \tag{3.28}$$

by adding terms proportional to the new constraints.

The first step in the proof consists in showing that the reduced phase space generators, i.e., the generators $Q_G(\omega, a)$ for which all variables except for the physical H^a_{TT} have been set to zero, are duality invariant. All other contributions to $Q_G(\omega, a)$ are then shown to be proportional to the constraints of Pauli-Fierz theory. Both these steps follow from straightforward but slightly tedious computations. For the generators of rotations and boosts for instance the computation is more involved because the explicit x^i dependence has to be taken into account when performing integrations by parts.

In terms of the new variables, the terms proportional to the constraints are bilinear in (h^1, A^2), (π^2, A^2), (h^1, C^1) and (π^2, C^1). The duality invariant generators $Q^D_G(\omega, a)$ are then obtained by adding the same terms with the substitution $h^1 \to h^2$, $A^2 \to -A^1$, $\pi^2 \to -\pi^1$ and $C^1 \to C^2$, while keeping unchanged the terms involving only the physical variables H^a_{TT}.

As a consequence, the duality invariant Poincaré transformations of h^1, π^2 are unchanged on the extended constraint surface. They are given by (A.31)-(A.32) where $\xi^\perp = -\omega^0{}_\nu x^\nu + a^0$ and $\xi^i = -\omega^i{}_\nu x^\nu + a^i$. Because of (3.28), those of $h^2, -\pi^1$ are obtained, on the contraint surface, by applying a duality rotation to the right hand-sides of (A.31)-(A.32).

An open question that we plan to address elsewhere is the construction of the canonical generators for global Poincaré transformations in the presence of both types of sources that will be introduced in the next section.

4 Coupling to conserved electric and magnetic sources

4.1 Interacting variational principle

We define

$$h^a_{0m} = n^a_m = h^a_{m0}, \qquad h^a_{00} = -2n^a, \tag{4.1}$$

and consider the action

$$S_T[z^A, u^\alpha; T^{a\mu\nu}] = \frac{1}{16\pi G} S_G + S^J, \tag{4.2}$$

with S_G given in (3.6) and the gauge invariant interaction term

$$S^J = \int d^4x \, \frac{1}{2} h^a_{\mu\nu} T^{\mu\nu}_a, \qquad \partial_\mu T^{\mu\nu}_a = 0, \tag{4.3}$$

where $T^{\mu\nu}_a \equiv (T^{\mu\nu}, \Theta^{\mu\nu})$ are external, conserved electric and magnetic energy-momentum tensors.

4.2 Linearized Taub-NUT solution

We start by considering the sources corresponding to a point-particle gravitational dyon with electric mass M and magnetic mass N at rest at the origin of the coordinate system, for which

$$T^{\mu\nu}_a(x) = \delta^\mu_0 \delta^\nu_0 M_a \delta^{(3)}(x^i), \qquad M_a = (M, N). \tag{4.4}$$

In this case, only the constraints (3.13) are affected by the interaction and become

$$\mathcal{H}_{a\perp} = -16\pi G M_a \delta^{(3)}(x). \tag{4.5}$$

They are solved by

$$\Delta C^a = G M^a (\frac{4}{r}), \tag{4.6}$$

where $r = \sqrt{x^i x_i}$. It is then straightforward to check that all equations of motions are solved by

$$C^a = GM^a(2r), \quad n^a = GM^a(-\frac{1}{r}), \quad A^a_m = n^{am} = H^a_{mn} = 0,$$
$$h^a_{mn} = GM^a(\delta_{mn} + \frac{x_m x_n}{r^3}), \quad \pi^{mn}_a = 0. \tag{4.7}$$

The usual Schwarzschild form is obtained after a gauge transformation with parameter $\xi^{am} = GM^a(-\frac{1}{2}\frac{x^m}{r})$, $\xi^{a\perp} = 0$. The solution then reads

$$C^a = GM^a(2r),\ n^a = GM^a(-\frac{1}{r}),\ A^a_m = GM^a(-\frac{1}{2}\frac{x_m}{r}),\ n^{am} = H^a_{mn} = 0,$$
$$h^a_{mn} = GM^a(\frac{2x_m x_n}{r^3}), \quad \pi^{mn}_a = 0. \tag{4.8}$$

By computing the Riemann tensor in terms of the canonical variables in Appendix **B**, we show that this solution describes the linearized Taub-NUT solution. It resolves the string singularity of the linearized Taub-NUT solution in the standard Pauli-Fierz formulation. In spherical coordinates, the latter can for instance be described by

$$h_{rr} = \frac{2GM}{r} = h_{00}, \qquad h_{0\varphi} = -2N(1 - \cos\theta), \tag{4.9}$$

and all other components vanishing, with a string-singularity along the negative z-axis.

5 Surface charges

5.1 Regge-Teitelboim revisited

Because the theory is not local as a Hamiltonian gauge theory, the analysis of surface charges cannot directly be performed as in [29, 30]. We thus revert to the original Hamiltonian method of [31, 32] and adapt it to the present situation of exact solutions, where there is no need to discuss fall-off conditions.

Let $\mathcal{L}_H = a_A \dot{z}^A - h - \gamma_\alpha u^\alpha$, with h a first class Hamiltonian density and γ_α first class constraints and define $\phi^i = (z^A, u^\alpha)$. Even though it is not so for our theory, let us first run through the arguments in the case where one has Darboux coordinates for the symplectic structure, i.e., when $\sigma_{AB} = \frac{\partial a_B}{\partial z^A} - \frac{\partial a_A}{\partial z^B}$ is the constant symplectic matrix. We furthermore suppose that we are in a source-free region of spacetime. In this case one can show that

$$\delta_\varepsilon z^A \frac{\delta \mathcal{L}_H}{\delta z^A} + \delta_\varepsilon u^\alpha \frac{\delta \mathcal{L}_H}{\delta u^\alpha} = -\partial_0\Big(\gamma_\alpha \varepsilon^\alpha\Big) - \partial_i s^i_\varepsilon, \tag{5.1}$$

where $s^i_\varepsilon = s^i_\varepsilon[z, u]$ vanishes when the Hamiltonian equations of motion, including constraints, are satisfied, $s^i_\varepsilon \approx 0$. This identity merely expresses the general fact that the

Noether current s^μ_ε associated to a gauge symmetry can be taken to vanish when the equations of motions hold (see e.g. [28], chapter 3), $s^\mu_\varepsilon \approx 0$, and that the integrand of the generator is given by (minus) the constraints contracted with the gauge parameters in the Hamiltonian formalism, $s^0_\varepsilon = -\gamma_\alpha \varepsilon^\alpha$. An explicit expression for s^i_ε in terms of the structure functions can for instance be found in Appendix D of [30]. Using integrations by parts, one can write the variations of the constraints under a change of the canonical coordinates z^A as an Euler-Lagrange derivative, up to a total derivative,

$$\delta_z(\gamma_\alpha\varepsilon^\alpha) = \delta z^A \frac{\delta(\gamma_\alpha\varepsilon^\alpha)}{\delta z^A} - \partial_i k^i_\varepsilon. \tag{5.2}$$

where $k^i_\varepsilon = k^i_\varepsilon[\delta z, z]$ depends linearly on δz^A and its spatial derivatives. Taking the time derivative of (5.2) and using a variation δ_ϕ of (5.1) to eliminate $\partial_0 \delta_z(\gamma_\alpha\varepsilon^\alpha)$, one finds

$$\partial_i\Big(\partial_0 k^i_\varepsilon - \delta_\phi s^i_\varepsilon\Big) = \partial_0\Big(\delta z^A \frac{\delta(\gamma_\alpha\varepsilon^\alpha)}{\delta z^A}\Big) + \delta_\phi\Big(\delta_\varepsilon z^A \frac{\delta \mathcal{L}_H}{\delta z^A} + \delta_\varepsilon u^\alpha \frac{\delta\mathcal{L}_H}{\delta u^\alpha}\Big). \tag{5.3}$$

One now takes ε^α_s to satisfy $\delta_{\varepsilon_s} z^A_s = 0 = \delta_{\varepsilon_s} u^\alpha_s$. Note that in the case of Darboux coordinates, this also implies that $\frac{\delta(\gamma_\alpha\varepsilon^\alpha_s)}{\delta z^A} = 0$. If furthermore z^A_s, u^α_s is a solution of the Hamiltonian equations of motion, the RHS of (5.3) also vanishes. By using a contracting homotopy with respect to $\delta\phi^i$ and their spatial derivatives, one deduces that

$$\partial_0 k^i_{\varepsilon_s}[\delta z, z_s] = (\delta_\phi s^i_{\varepsilon_s})[z_s] - \partial_j k^{[ij]}_{\varepsilon_s}, \tag{5.4}$$

where $k^{[ij]}_{\varepsilon_s} = k^{[ij]}_{\varepsilon_s}[\delta\phi, \phi_s]$ depends linearly on $\delta\phi^i$ and their spatial derivatives. Finally, when $\delta z^A_s, \delta u^\alpha_s$ satisfy the linearized Hamiltonian equations of motion, including constraints, we find from (5.2) and (5.4) that

$$\partial_i k^i_{\varepsilon_s}[\delta z_s, z_s] = 0, \quad \partial_0 k^i_{\varepsilon_s}[\delta z_s, z_s] - \partial_j t^{[ij]}_{\varepsilon_s} = 0. \tag{5.5}$$

At a fixed time $t = x^0$, consider a closed 2 dimensional surface S, $\partial S = 0$, for instance a sphere with radius r and define the surface charge 1-forms by

$$\not{\delta}\mathcal{Q}_{\varepsilon_s}[\delta z_s, z_s] = \oint_S d^2x_i\, k^i_{\varepsilon_s}[\delta z_s, z_s], \tag{5.6}$$

where $d^2x_i = \frac{1}{2}\epsilon_{ijk} dx^j \wedge dx^k$. The first relation of (5.5) implies that the surface charge 1-form only depends on the homology class of the closed surface S,

$$\oint_{S_1} d^2x_m\, k^m_{\varepsilon_s}[\delta z_s, z_s] = \oint_{S_2} d^2x_m\, k^m_{\varepsilon_s}[\delta z_s, z_s]. \tag{5.7}$$

Here $S_1 - S_2 = \partial\Sigma$, where Σ is a three-dimensional volume at fixed time t containing no sources. For instance, the surface charge 1-form does not depend on r. The second relation of (5.5) implies that it is conserved in time and so does not depend on t either,

$$\frac{d}{dt}\not{\delta}\mathcal{Q}_{\varepsilon_s}[\delta z_s, z_s] = 0. \tag{5.8}$$

The question is then whether these charge 1-forms are integrable, see e.g. [33, 34, 30] for a discussion.

5.2 Linear theories

In the case of linear theories, the latter problem does not arise and the whole analysis simplifies. One can replace (5.2) by

$$\gamma_\alpha \varepsilon^\alpha = z^A \frac{\delta(\gamma_\alpha \varepsilon^\alpha)}{\delta z^A} - \partial_i k^i_\varepsilon[z], \tag{5.9}$$

where $\delta/\delta z^A$ are the (spatial) Euler-Lagrange derivatives and $k^i_\varepsilon[z]$ depends linearily both on the phase space variables z^A and their spatial derivatives and on the gauge parameters. One then uses (5.1) directly to eliminate $\partial_0(\gamma_\alpha\varepsilon^\alpha)$ from the time derivative of (5.9), to get

$$\partial_i\Big[\partial_0 k^i_\varepsilon - s^i_\varepsilon\Big] = \partial_0\Big[z^A \frac{\delta(\gamma_\alpha\varepsilon^\alpha)}{\delta z^A}\Big] + \delta_\varepsilon z^A \frac{\delta \mathcal{L}_H}{\delta z^A} + \delta_\varepsilon u^\alpha \frac{\delta \mathcal{L}_H}{\delta u^\alpha}. \tag{5.10}$$

For gauge parameters ε^α_s that satisfy

$$\delta_{\varepsilon_s} z^A = 0 = \delta_{\varepsilon_s} u^\alpha, \tag{5.11}$$

one then arrives at

$$\partial_i k^i_{\varepsilon_s}[z] = -\gamma_\alpha \varepsilon^\alpha, \quad \partial_0 k^i_{\varepsilon_s}[z] = s^i_{\varepsilon_s}[z,u] - \partial_j k^{[ij]}_{\varepsilon_s}. \tag{5.12}$$

For a solution z^a_s, u^α_s, the surface charges

$$\mathcal{Q}_{\varepsilon_s}[z_s] = \oint_S d^2x_i\, k^i_{\varepsilon_s}[z_s], \tag{5.13}$$

are again independent of r and t.

When this analysis is applied to the Hamiltonian formulation of Pauli-Fierz theory, one finds the standard expressions

$$k^i_\varepsilon[z] = 2\xi_m \pi^{mi} - \xi^\perp(\delta^{mn}\partial^i - \delta^{mi}\partial^n)h_{mn} + h_{mn}(\delta^{mn}\partial^i - \delta^{ni}\partial^m)\xi^\perp, \tag{5.14}$$

while the only solutions to (5.11) are $\xi_{\mu s} = -\omega_{[\mu\nu]}x^\nu + a_\mu$, for some constants a_μ, $\omega_{[\mu\nu]} = -\omega_{[\nu\mu]}$. In this context of flat space, Greek indices take values from 0 to 3 with $\mu = (\perp, i)$. Indices μ are lowered and raised with $\eta_{\mu\nu} = \text{diag}\,(-1,1,1,1)$.

5.3 Electric and magnetic energy-momentum and angular momentum surface charges

The previous analysis is not directly applicable to our case since we do not have Darboux coordinates and the Poisson brackets of the fundamental variables are non-local. In particular, the gauge transformations (3.17) do not allow for non trivial solutions to $\delta_{\varepsilon_s} z^A = 0$. We also have to keep the sources explicitly throughout the argument, because

Δ^{-1} applied to localized sources will spread them out throughout space and we need to check that we are only dropping terms that indeed vanish outside of the sources.

In the presence of the sources, the constraints $\gamma^J_\alpha = (\mathcal{H}^J_{am}, \mathcal{H}^J_{a\perp})$ are determined

$$\mathcal{H}^J_{am} = \mathcal{H}_{am} - (16\pi G)T^0_{am}, \quad \mathcal{H}^J_{a\perp} = \mathcal{H}_{a\perp} - (16\pi G)T^0_{a0}. \tag{5.15}$$

Instead of (5.9), we can write

$$\gamma^J_\alpha \varepsilon^\alpha = (\partial^m \xi^{an} + \partial^n \xi^{am})\epsilon_{ab}\pi^b_{mn} + (\delta^{mn}\Delta - \partial^m\partial^n)\xi^{a\perp}h_{amn} - \partial_i \widetilde{k}^i_\varepsilon[z] \\ - (16\pi G)(T^0_{am}\xi^{am} + T^0_{a0}\xi^{a\perp}), \tag{5.16}$$

where

$$\widetilde{k}^i_\varepsilon[z] = 2\xi^a_m \epsilon_{ab}\pi^{bmi} - \xi^{a\perp}(\delta^{mn}\partial^i - \delta^{mi}\partial^n)h_{amn} + h_{amn}(\delta^{mn}\partial^i - \delta^{ni}\partial^m)\xi^{a\perp}. \tag{5.17}$$

Consider now gauge parameters $\epsilon^\alpha_s(x)$ satisfying the conditions

$$\begin{cases} \partial^m \xi^{an}_s + \partial^n \xi^{am}_s = 0 = \partial_0 \xi^{am}_s - \partial^m \xi^{a\perp}_s, \\ (\delta^{mn}\Delta - \partial^m\partial^n)\xi^{a\perp}_s = 0 = \partial_0 \xi^{a\perp}_s, \end{cases} \tag{5.18}$$

The general solution to conditions (5.18) can be written as

$$\xi^a_{\mu s} = -\omega^a_{[\mu\nu]}x^\nu + a^a_\mu, \tag{5.19}$$

for some constants a^a_μ, $\omega^a_{[\mu\nu]} = -\omega^a_{[\nu\mu]}$. It follows in particular that the surface charges

$$\mathcal{Q}_{\varepsilon_s}[z_s] = \frac{1}{16\pi G}\oint_S d^2x_i\, \widetilde{k}^i_{\varepsilon_s}[z_s], \tag{5.20}$$

do not depend on the homology class of S outside of sources.

Assuming Δ invertible, the equations of motion associated to $\mathcal{L}_T = \frac{1}{16\pi G}\mathcal{L}_H + \mathcal{L}^J$ imply in particular that

$$\partial_0 h^a_{mn} = \partial_m n^a_n + \partial_n n^a_m - 2\epsilon^{ab}\Delta H_{bmn} + \epsilon^{ab}\delta_{mn}\Delta H_b \\ + (16\pi G)\epsilon^{ab}\Big(\Delta^{-1}(\mathcal{O}T_b)_{mn} + \frac{1}{2}\Delta^{-2}\partial_m\epsilon_{npq}\partial^p\partial_k T^{kq}_b + \frac{1}{2}\Delta^{-2}\partial_n\epsilon_{mpq}\partial^p\partial_k T^{kq}_b\Big), \tag{5.21}$$

$$\epsilon_{ab}\partial_0\pi^b_{mn} = \left(\mathcal{P}^{TT}H_a\right)_{mn} + (8\pi G)T_{amn} - \\ - \frac{1}{2}(\delta^{mn}\Delta - \partial^m\partial^n)(2n_a + \frac{1}{2}\Delta C_a). \tag{5.22}$$

By direct computation using the equations of motion, one then finds

$$\partial_0 \widetilde{k}^i_{\varepsilon_s}[z_s] = (16\pi G)(\xi^a_{\mu s}T^{\mu i}_a) - \partial_j k^{[ij]}_{\varepsilon_s}[z_s, u_s], \tag{5.23}$$

with

$$
\begin{aligned}
k^{[ij]}_{\varepsilon_s}[z,u] = \Big(2n^i_a\partial^j\xi^{a\perp}_s + \xi^{a\perp}_s\partial^i n^j_a + \xi^{ai}_s\partial^j(2n_a + \frac{1}{2}\Delta C_a) + \xi^a_{sm}\epsilon^{mpq}\partial_p\partial^i H^j_{aq} \\
+ \omega^{aj}\partial^k H^i_{ak} + \omega^{ai}\partial^j H_a + 2\omega^{ak}\partial^i H^j_{ak} + 16\pi G\epsilon^{ab}\epsilon^{imq}\Delta^{-1}T^j_{bq}\partial_m\xi^\perp_{as} \\
+ 8\pi G\epsilon^{ab}\epsilon^{mpq}\partial_p\Delta^{-2}\partial^i T^j_{bq}\partial_m\xi^\perp_{as} - (i \longleftrightarrow j)\Big) \\
+ \epsilon^{ijk}\Big[\omega^a_k(2n_a + \frac{1}{2}\Delta C_a) - \xi^{am}_s(\Delta H_{amk} - \partial_m\partial^r H_{ark}) \\
- 16\pi G\epsilon^{ab}\Delta^{-1}\partial^r T_{brk}\xi^\perp_{as} + 8\pi G\epsilon^{ab}(\Delta^{-1}T^m_{bk} + \Delta^{-2}\partial^m\partial^r T_{brk})\partial_m\xi^\perp_{as}\Big],
\end{aligned}
\tag{5.24}
$$

where $\omega^a_{mn} = \omega^{ak}\epsilon_{kmn}$. The surfaces charges (5.20) are thus also time-independent outside of sources.

Finally, the surface charges are gauge invariant,

$$\widetilde{k}^i_{\varepsilon_s}[\delta_\eta z] = \partial_j r^{[ij]}_{\varepsilon_s,\eta}, \tag{5.25}$$

$$r^{[ij]}_{\varepsilon_s,\eta} = \Big(2\xi^{aj}_s\partial^i\eta^\perp_a + 2\eta^j_a\partial^i\xi^{a\perp}_s + \xi^{a\perp}_s\partial^j\eta^i_a - (i \longleftrightarrow j)\Big) - 2\epsilon^{ijk}\omega^a_k\eta^\perp_a. \tag{5.26}$$

Defining

$$Q_{\epsilon_s}[z] = \frac{1}{2}\omega^a_{\mu\nu}J^{\mu\nu}_a - a^a_\mu P^\mu_a, \tag{5.27}$$

we get for the individual generators

$$
\begin{aligned}
(16\pi G)P^\perp_a &= -\oint_{S^\infty} d^2x_m\,\partial^m\Delta C_a = \oint_{S^\infty} d^2x_m\,(\partial_n h^{mn}_a - \partial^m h_a)\,, && (5.28)\\
(16\pi G)P^n_a &= 2\oint_{S^\infty} d^2x_m\,\epsilon_{ab}\Delta H^{bnm} = -2\oint_{S^\infty} d^2x_m\,\epsilon_{ab}\pi^{bmn}, && (5.29)\\
(16\pi G)J^{kl}_a &= 2\oint_{S^\infty} d^2x_m\,\epsilon_{ab}\left(\Delta H^{bmk}x^l - \Delta H^{bml}x^k\right) && (5.30)\\
&= -2\oint_{S^\infty} d^2x_m\,\epsilon_{ab}\left(\pi^{bmk}x^l - \pi^{bml}x^k\right), && (5.31)\\
(16\pi G)J^{\perp k}_a &= \oint_{S^\infty} d^2x_m\,\left(\Delta C^a\delta^{mk} - \partial^m\Delta C_a x^k\right) && (5.32)\\
&= \oint_{S^\infty} d^2x_m\,\left[(\partial_n h^{mn}_a - \partial^m h_a)\,x^k - h^{mk}_a + h_a\delta^{mk}\right]. && (5.33)
\end{aligned}
$$

The only non-vanishing surface charges of the dyon sitting at the origin are

$$P^\perp_a = M_a. \tag{5.34}$$

As expected, they measure the electric and magnetic mass of the dyon.

For later use, we combine $\widetilde{k}^i_\varepsilon, k^{[ij]}_\varepsilon$ into the $n-2$ forms $k_\varepsilon[z,u]$ through the following expressions in Cartesian coordinates,

$$k_\varepsilon = k^{[\mu\nu]}_\varepsilon d^{n-2}x_{\mu\nu}, \qquad k^{[0i]}_\varepsilon = \widetilde{k}^i_\varepsilon, \tag{5.35}$$

$$d^{n-k}x_{\mu_1\ldots\mu_k} = \frac{1}{k!(4-k)!}\epsilon_{\mu_1\ldots\mu_k\nu_{k+1}\ldots\nu_4}dx^{\nu_{k+1}}\ldots dx^{\nu_4}, \tag{5.36}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is completely skew-symmetric with $\epsilon_{0123} = 1$ and the wedge product between the differentials is understood. Equations (5.16) and (5.23) can then be summarized by

$$dk_{\varepsilon_s} \approx -(16\pi G)T_{\varepsilon_s}, \quad T_{\varepsilon_s} = T^{\mu}_{a\nu}\xi^{a\nu}_{s} d^3x_{\mu}, \quad dT_{\varepsilon_s} = 0, \tag{5.37}$$

where closure of the $n-1$-forms T_{ε_s} follows from the conservation of the sources, the symmetry of the energy-momentum tensor and (5.19).

Remark: In fact we have checked here that the standard expressions for surface charges in Pauli-Fierz theory, when extended in a duality invariant way, have all the expected properties. More interesting would be to develop the theory of surface charges from scratch in theories where the Poisson brackets among the fundamental variables are not local to see if the ones we have found exhaust all possibilities. From the preceding discussion we see that pseudo-differential operators will play a crucial role for a discussion of these generalized conservation laws, as they do in the discussion of ordinary conservation laws for evolution equations of the Korteweg-de Vries type for instance.

5.4 Poincaré transformations of surface charges

Suppose now that z^A_s, u^α_s solve the equations of motions for the conserved sources $T^{\mu\nu}_a(x)$. Let z'^A_s, u'^α_s be the solution associated to new sources $T'^{\mu\nu}_a(x)$ related to $T^{\mu\nu}_a(x)$ through a (proper) Poincaré transformation, $x'^\mu = \Lambda^\mu{}_\nu x^\nu + b^\mu$ with $|\Lambda| = 1$,

$$T'^{\mu\nu}_a(x') = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta}_a(x). \tag{5.38}$$

For instance, starting from the conserved energy-momentum tensors (4.4) of a dyon sitting at the origin with world-line $z^\mu = \delta^\mu_0 s$, one can obtain in this way the conserved energy-momentum tensors of a dyon moving along a straight line, $z'^\mu = u^\mu s + a^\mu$ with u^μ, a^μ constant, $u^\mu u_\mu = -1$ and s the proper time,

$$T'_a{}^{\mu\nu}(x') = M_a u^\nu \int d\lambda \delta^{(4)}(x' - z'(\lambda)) \frac{dz'^\mu}{d\lambda} = M_a \frac{u^\mu u^\nu}{u^0} \delta^{(3)}(x'^i - z'^i(x^0)). \tag{5.39}$$

when $\Lambda^\mu{}_0 = u^\mu$.

Assume then that the $\xi^\mu_{as}(x)$ transform like vectors

$$\xi'^\nu_{as}(x') = \Lambda^\mu{}_\alpha \xi^\alpha_{as}(x) = -(\Lambda\omega_a\Lambda^{-1}x')^\nu + (\Lambda\omega_a\Lambda^{-1}b + \Lambda a_a)^\nu, \tag{5.40}$$

which implies that the T_{ε_s} are closed Poincaré invariant $n-1$ forms,

$$T'_{\varepsilon'_s}(x', dx') = T_{\varepsilon_s}(x, dx). \tag{5.41}$$

We can then use the following variant of the tube lemma. Suppose that at fixed time x^0, $T^{0\nu}_a(x)\xi^{a\nu}_s(x)$ has compact support and that there exists a tube, i.e., a space-time volume

$\mathcal{W}$ connecting the hypersurfaces $\Omega : K = x^0$ and $\Omega' : K = x'^0 = \Lambda^0{}_\nu x^\nu + b^0$ with K a constant such that, $\partial\mathcal{W} = \Omega' - \Omega + \mathcal{T}$. If nothing flows out through $\mathcal{T}$, $\int_{\mathcal{T}} T_{\varepsilon_s} = 0$, it follows from Stokes' theorem and (5.41) that

$$\int_\Omega T_{\varepsilon_s} = \int_{\Omega'} T_{\varepsilon_s} = \int_{\Omega'} T'_{\varepsilon'_s}. \tag{5.42}$$

If we now compute the surface charges for a large enough sphere S at fixed x^0 containing both $T^{0\nu}_a(x)\xi^{a\nu}_s(x)$ and $T'^{0\nu}_a(x)\xi'^{a\nu}_s(x)$, it finally follows from (5.16) or (5.37) that the surface charges evaluated for the new solutions z'^A are obtained from those of the old solutions z^A through

$$\mathcal{Q}_{\varepsilon'_s}[z'_s] = \mathcal{Q}_\varepsilon[z_s]. \tag{5.43}$$

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A Poincaré generators for Pauli-Fierz theory

In this appendix, we assume that the canonical variables vanish sufficiently fast at the boundary so that integrations by parts can be used even if the gauge parameters do not vanish at the boundary.

In the Hamiltonian formulation of general relativity [25], the canonically conjugate variables are the spatial 3 metric g_{ij} and the extrinsic curvature π^{ij}. The constraints are explicitly given by

$$\mathcal{H}_\perp = \frac{1}{\sqrt{g}}\left(\pi^{mn}\pi_{mn} - \frac{1}{2}\pi^2\right) - \sqrt{g}R\,, \qquad \mathcal{H}_i = -2\nabla_j \pi^j{}_i. \tag{A.1}$$

The associated generators of gauge transformation $H[\xi] = \int d^3x\, \left(\mathcal{H}_\perp\xi^\perp + \mathcal{H}_i\xi^i\right)$ satisfy the so-called surface deformation algebra [35, 36],

$$\begin{aligned}
\{H[\xi], H[\eta]\} &= H[[\xi,\eta]_{SD}], && (A.2)\\
[\xi,\eta]^\perp_{SD} &= \xi^i\partial_i\eta^\perp - \eta^i\partial_i\xi^\perp, && (A.3)\\
[\xi,\eta]^i_{SD} &= g^{ij}\left(\xi^\perp\partial_j\eta^\perp - \eta^\perp\partial_j\xi^\perp\right) + \xi^j\partial_j\eta^i - \eta^j\partial_j\xi^i. && (A.4)
\end{aligned}$$

When the parameters f, g of gauge transformations depend on the canonical variables, (A.2) is replaced by [37]

$$\{H[f], H[g]\} = H[k], \tag{A.5}$$

$$k = [f, g]_{SD} + \delta_g f - \delta_f g - m, \tag{A.6}$$

$$m^{\perp} = \int d^3x' \Big[\{f^{\perp}, g^{\perp}(x')\}\mathcal{H}_{\perp}(x') + \{f^{\perp}, g^j(x')\}\mathcal{H}_j(x')\Big], \tag{A.7}$$

$$m^{i} = \int d^3x' \Big[\{f^{i}, g^{\perp}(x')\}\mathcal{H}_{\perp}(x') + \{f^{i}, g^j(x')\}\mathcal{H}_j(x')\Big], \tag{A.8}$$

where

$$\delta_\xi g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i + 2D_{ijkl}\pi^{kl}\xi^{\perp}, \tag{A.9}$$

$$D_{ijkl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{jk}g_{il} - g_{ij}g_{kl}), \tag{A.10}$$

$$\begin{aligned}\delta_\xi \pi^{ij} &= -\xi^{\perp}\sqrt{g}(R^{ij} - \frac{1}{2}g^{ij}R) + \frac{\xi^{\perp}}{2\sqrt{g}}g^{ij}(\pi^{kl}\pi_{kl} - \frac{1}{2}\pi^2)\\ &\quad - 2\frac{\xi^{\perp}}{\sqrt{g}}(\pi^{im}\pi^j_m - \frac{1}{2}\pi^{ij}\pi) + \sqrt{g}(\nabla^j\nabla^i\xi^{\perp} - g^{ij}\nabla_m\nabla^m\xi^{\perp})\\ &\quad + \nabla_m(\pi^{ij}\xi^m) - \nabla_m\xi^i\pi^{mj} - \nabla_m\xi^j\pi^{mi}.\end{aligned} \tag{A.11}$$

Let $g_{ij} = \delta_{ij} + h_{ij}$ and consider the canonical change of variables from g_{ij}, π^{kl} to $z^A = (h_{ij}, \pi^{kl})$. We will expand in terms of the homogeneity in the new variables and use the flat metric δ_{ij} to raise and lower indices in the remainder of this appendix. Furthermore, Greek indices take values from 0 to 3 with $\mu = (\perp, i)$. Indices are lowered and raised with $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ and its inverse. Let $\widetilde{\omega}_{\mu\nu} = -\widetilde{\omega}_{\nu\mu}$.

To lowest order, i.e., when $g_{ij} = \delta_{ij}$, the vector fields

$$\xi_P(\widetilde{\omega}, \widetilde{a})^{\mu} = -\widetilde{\omega}^{\mu}{}_i x^i + \widetilde{a}^{\mu}, \tag{A.12}$$

equipped with the surface deformation bracket form a representation of the Poincaré algebra [31],

$$[\xi_P(\widetilde{\omega}_1, \widetilde{a}_1), \xi_P(\widetilde{\omega}_2, \widetilde{a}_2)]^{(0)}_{SD} = \xi_P([\widetilde{\omega}_1, \widetilde{\omega}_2], \widetilde{\omega}_1\widetilde{a}_2 - \widetilde{\omega}_2\widetilde{a}_1). \tag{A.13}$$

For the gauge generators, we find $H[\xi] = H^{(1)}[\xi] + H^{(2)}[\xi] + H^{(3)}[\xi] + \cdots$, where

$$H^{(1)}[\xi] = \int d^3x\, \left(-2\partial^j\pi_{ij}\xi^i + (\partial^i\partial^j h_{ij} - \Delta h)\xi^{\perp}\right) \tag{A.14}$$

$$= \int d^3x\, \left(\mathcal{H}^{(1)}_m\xi^m + \mathcal{H}^{(1)}_{\perp}\xi^{\perp}\right) \tag{A.15}$$

are the gauge generators associated to the constraints (2.3) of the Pauli-Fierz theory. Because

$$H[[\xi, \eta]_{SD}] = H^{(1)}[[\xi, \eta]^{(0)}_{SD}] + H^{(2)}[[\xi, \eta]^{(0)}_{SD}] + H^{(1)}[[\xi, \eta]^{(1)}_{SD}] + O(z^3), \tag{A.16}$$

we have to lowest non trivial order

$$\big\{H^{(1)}[\xi],H^{(2)}[\eta]\big\}=H^{(1)}[[\xi,\eta]^{(0)}_{SD}]. \tag{A.17}$$

This means that $H^{(2)}[\eta]$ are observables, i.e., weakly gauge invariant functionals.

One can use integrations by parts to show that $H^{(1)}[\xi_P]=0$. It then follows that

$$\{H[\xi_P],H[\eta_P]\}=\big\{H^{(2)}[\xi_P],H^{(2)}[\eta_P]\big\}+O(z^3). \tag{A.18}$$

For vectors $\xi_P(\widetilde{\omega},\widetilde{a}),\eta_P(\widetilde{\theta},\widetilde{b})$ of the form (A.12), the first term on the RHS of (A.16) vanishes on account of (A.13). To lowest non trivial order, (A.2) then implies

$$\big\{H^{(2)}[\xi_P],H^{(2)}[\eta_P]\big\}=H^{(2)}[[\xi_P,\eta_P]^{(0)}_{SD}]+H^{(1)}[[\xi_P,\eta_P]^{(1)}_{SD}]. \tag{A.19}$$

The generators $H^{(2)}[\xi_P]$ equipped with the Poisson bracket thus form a representation of the Poincaré algebra when the constraints of the Pauli-Fierz theory are satisfied. Explicitly, the term proportional to the constraints is

$$H^{(1)}[[\xi,\eta]^{(1)}_{SD}]=-2\int d^3x\,\partial^j\pi_{ji}h^{ik}(\xi^\perp_P\theta^\perp{}_k-\eta^\perp_P\omega^\perp{}_k), \tag{A.20}$$

while

$$\mathcal{H}^{(2)}_i=-2\partial_j\left(\pi^{jk}h_{ik}\right)+\pi^{jk}\partial_ih_{jk} \tag{A.21}$$

$$\begin{aligned}\mathcal{H}^{(2)}_\perp&=\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^2\\&+\frac{1}{4}\partial_kh_{ij}\partial^kh^{ij}-\frac{1}{2}\partial_kh^{ki}\partial^jh_{ij}+\frac{1}{2}\partial_ih\partial_jh^{ij}-\frac{1}{4}\partial_ih\partial^ih\\&+\partial_l\left(\frac{1}{2}h\partial^lh-h^{ij}\partial^lh_{ij}-\frac{1}{2}h\partial_ih^{il}-h^{il}\partial_ih+\frac{3}{2}h^{lj}\partial^ih_{ij}+\frac{1}{2}h_{ij}\partial^ih^{jl}\right).\end{aligned} \tag{A.22}$$

Isolating terms proportional to the constraints, we find

$$H^{(2)}[\xi]=\int d^3x\left(\mathcal{H}_mh^{mi}\xi_i+\frac{1}{2}\mathcal{H}h\xi^\perp\right)+\bar{H}^{(2)}[\xi], \tag{A.23}$$

$$\bar{\mathcal{H}}^{(2)}_i=-\pi^{jk}(\partial_jh_{ki}+\partial_kh_{ji}-\partial_ih_{jk}), \tag{A.24}$$

$$\begin{aligned}\bar{\mathcal{H}}^{(2)}_\perp&=\pi^{ij}\pi_{ij}-\frac{1}{2}\pi^2\\&+\frac{1}{4}\partial_kh_{ij}\partial^kh^{ij}-\frac{1}{2}\partial_kh^{ki}\partial^jh_{ij}+\frac{1}{4}\partial_ih\partial^ih\\&+\partial_l\left(-h^{ij}\partial^lh_{ij}-h^{il}\partial_ih+\frac{3}{2}h^{lj}\partial^ih_{ij}+\frac{1}{2}h_{ij}\partial^ih^{jl}\right),\end{aligned} \tag{A.25}$$

with $\bar{H}^{(2)}[\xi]=\int d^3x\left(\bar{\mathcal{H}}^{(2)}_i\xi^i+\bar{\mathcal{H}}^{(2)}_\perp\xi^\perp\right)$. On account of (A.17) and the analog of (A.5) for $H^{(1)}[f]$, it follows that

$$\big\{\bar{H}^{(2)}[\xi_P],\bar{H}^{(2)}[\eta_P]\big\}\approx\bar{H}^{(2)}[[\xi_P,\eta_P]^{(0)}_{SD}], \tag{A.26}$$

where $\approx$ means an equality up to terms proportional to the constraints $\mathcal{H}_m, \mathcal{H}_\perp$ of Pauli-Fierz theory. Note that the functionals $H^{(2)}[\xi_P]$ and $\bar{H}^{(2)}[\xi_P]$ generate transformations of the canonical variables that are equivalent because they differ at most by a gauge transformation of the Pauli-Fierz theory when restricted to the constraint surface.

The generators for global Poincaré transformations of Pauli-Fierz theory can then be identified as

$$\begin{aligned} &Q_G(\omega, a) = \frac{1}{2}\omega_{\mu\nu} J_G^{\mu\nu} - a_\mu P_G^\mu = \bar{H}^{(2)}[\xi_P(\widetilde{\omega}, \widetilde{a})] \\ &\widetilde{\omega}_{\mu\nu} = \omega_{\mu\nu}, \quad \widetilde{a}_\perp = a_\perp, \quad \widetilde{a}_i = a_i + \omega_{\perp i} x^0. \end{aligned} \tag{A.27}$$

Indeed, differentiating (A.19) with respect to $b_\perp$ gives

$$\{H, Q_G(\omega, a)\} = \frac{\partial}{\partial t} Q_G(\omega, a) + 2\int d^3x\, \partial^j \pi_{ji} h^{ik} \omega_{\perp k}. \tag{A.28}$$

When combined with (A.19) and (A.27), this shows that, on the constraint surface, the generators $Q_G(\omega, a)$ are conserved and satisfy the Poincaré algebra.

Finally, we can further simplify the explicit expression for $\bar{H}^{(2)}[\xi_P]$ by using linearity of ξ_P in x^i and integrations by parts to show that

$$\begin{aligned} \int d^3x\, \bar{\mathcal{H}}_\perp^{(2)} \xi_P^\perp = \int d^3x \Big[&\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2 + \frac{1}{4}\partial_k h_{ij} \partial^k h^{ij} - \frac{1}{2}\partial_k h^{ki} \partial^j h_{ij} \\ &+ \frac{1}{4}\partial_i h \partial^i h + \partial_l \big(h \partial_i h^{il} + h^{lj} \partial^i h_{ij} \big) \Big] \xi_P^\perp. \end{aligned} \tag{A.29}$$

The expansion of the gauge transformations (A.9), (A.11) gives to first order:

$$\delta_\xi^{(0)} h_{ij} = \partial_i \xi_j + \partial_j \xi_j, \qquad \delta_\xi^{(0)} \pi^{ij} = (\partial^i \partial^j - \delta^{ij}\Delta)\xi^\perp, \tag{A.30}$$

$$\delta_\xi^{(1)} h_{ij} = \xi^k \partial_k h_{ij} + \partial_i \xi^k h_{kj} + \partial_j \xi^k h_{ik} + 2\pi_{ij}\xi^\perp - \delta_{ij}\pi\xi^\perp, \tag{A.31}$$

$$\begin{aligned} \delta_\xi^{(1)} \pi^{ij} = &\frac{1}{2} h (\partial^i \partial^j - \delta^{ij}\Delta)\xi^\perp - h^{im}\partial_m \partial^j \xi^\perp - h^{jm}\partial_m \partial^i \xi^\perp + h^{ij}\Delta\xi^\perp \\ &+ \delta^{ij} h^{mn} \partial_m \partial_n \xi^\perp + \partial_m(\pi^{ij}\xi^m) - \pi^{mj}\partial_m \xi^i - \pi^{mi}\partial_m \xi^j \\ &+ \frac{1}{2}\partial_k \xi^\perp \Big[-\partial^j h^{ki} - \partial^i h^{kj} + \partial^k h^{ij} + \delta^{ij}(2\partial_l h^{kl} - \partial^k h) \Big] \\ &+ \frac{1}{2}\xi^\perp \Big[\partial^i \partial^j h + \Delta h^{ij} - \partial_k \partial^i h^{jk} - \partial_k \partial^j h^{ik} - \delta^{ij}(\Delta h - \partial_k \partial_l h^{kl}) \Big]. \end{aligned} \tag{A.32}$$

B Riemann tensor and canonical variables

By following [1, 11] (up to conventions), we show in this appendix that the duality rotations that we have defined coincide on-shell with the standard simultaneous duality rotations among the (linearized) Riemann tensor and its dual together with those for the

electric and magnetic conserved sources. We do this by showing how the covariant Riemann tensor is expressed in terms of the canonical variables. This gives us the appropriate generalization of the Gauss-Codazzi relations in the case of both electric and magnetic sources.

B.1 Covariant equations in the presence of magnetic sources

Our conventions are as follows. Define $\epsilon_{a_1\dots a_n} = \epsilon^{a_1\dots a_n}$ to be totally skew-symmetric with $\epsilon_{1\dots n} = 1$. The Levi-Civita tensor is $\boldsymbol{\varepsilon}_{a_1\dots a_n} = \sqrt{|g|}\epsilon_{a_1\dots a_n}$. Indices on this tensor are raised with the metric, which implies that $\boldsymbol{\varepsilon}^{a_1\dots a_n} = \frac{(-)^\sigma}{\sqrt{|g|}}\epsilon^{a_1\dots a_n}$ where σ is the signature of the metric. Our convention for the dual is ${}^*\omega_{a_1\dots a_{n-p}} = \frac{1}{p!}\omega^{b_1\dots b_p}\boldsymbol{\varepsilon}_{b_1\dots b_p a_1\dots a_{n-p}}$.

In flat Minkowski spacetime, traces are taken with the flat Minkowsi metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Start with the "(linearized) Riemann tensor" $R^1_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma}$, with only symmetry properties skew-symmetry in the first and last pairs of indices,

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}. \tag{B.1}$$

Its double dual (cf. MTW [38])

$$\not{G}^1_{\mu\nu\rho\sigma} \equiv \not{G}_{\mu\nu\rho\sigma} = \frac{1}{4}\boldsymbol{\varepsilon}_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\gamma\delta}\boldsymbol{\varepsilon}_{\gamma\delta\rho\sigma}, \tag{B.2}$$

has the same symmetry properties. If ordinary duals of $R_{\mu\nu\rho\sigma}$ and $\not{G}_{\mu\nu\rho\sigma}$ are taken with respect to the last pair of indices, and we define

$$R^2_{\mu\nu\rho\sigma} \equiv -{}^*R_{\mu\nu\rho\sigma} = -\frac{1}{2}R_{\mu\nu}{}^{\alpha\beta}\boldsymbol{\varepsilon}_{\alpha\beta\rho\sigma}, \tag{B.3}$$

$$\not{G}^2_{\mu\nu\rho\sigma} \equiv -{}^*\not{G}_{\mu\nu\rho\sigma} = \frac{1}{2}\boldsymbol{\varepsilon}_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}, \tag{B.4}$$

we have

$$R^a_{\mu\nu\rho\sigma} = \epsilon^{ab}({}^*R)_{b\,\mu\nu\rho\sigma}, \quad \not{G}^a_{\mu\nu\rho\sigma} = \epsilon^{ab*}\not{G}_{b\,\mu\nu\rho\sigma}. \tag{B.5}$$

The 36 independent components of the Riemann tensor can be encoded in

$$R^a_{0m0n}, \quad \not{G}^a_{0m0n}. \tag{B.6}$$

The Ricci and Einstein tensors are defined as

$$R^a_{\mu\nu} = R^{a\alpha}{}_{\mu\alpha\nu}, \quad G^a_{\mu\nu} = \not{G}^{a\alpha}{}_{\mu\alpha\nu} = R^a_{\nu\mu} - \frac{1}{2}\eta_{\mu\nu}R^a. \tag{B.7}$$

If electric and magnetic conserved sources are $T^{\mu\nu}_a \equiv (T^{\mu\nu}, \Theta^{\mu\nu})$, with $T^{\mu\nu}_a = T^{\nu\mu}_a$ symmetric, $\partial_\mu T^{\mu\nu}_a = 0$, the duality rotations are defined by

$$R^{a\prime}_{\mu\nu\rho\sigma} = M^a{}_b R^b_{\mu\nu\rho\sigma}, \qquad T^{a\prime}_{\mu\nu} = M^a{}_b T^b_{\mu\nu},$$
$$M_{ca}M^c{}_b = \delta_{ab}. \tag{B.8}$$

For a tensor $K^{\mu\nu}$, let $\bar{K}^{\mu\nu} = K^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}K$.

It has been shown in [11] that the duality invariant equations of motion are

$$G_a^{\mu\nu} = 8\pi G\, T_a^{\mu\nu} \iff R^a_{\mu\nu\rho\sigma} + R^a_{\mu\sigma\nu\rho} + R^a_{\mu\rho\sigma\nu} = 8\pi G\, \epsilon^{ab}\boldsymbol{\varepsilon}_{\delta\nu\rho\sigma}\, \overline{T}^{\delta}_{\ b\,\mu}. \tag{B.9}$$

They imply in particular that, on-shell, the tensors $R^a_{\mu\nu\rho\sigma}$ are symmetric in the exchange of pairs of indices and that $R^a_{\mu\nu}, G^a_{\mu\nu}$ are symmetric. Furthermore, the Bianchi "identities" read

$$\partial_\epsilon R^a_{\gamma\delta\alpha\beta} + \partial_\beta R^a_{\gamma\delta\epsilon\alpha} + \partial_\alpha R^a_{\gamma\delta\beta\epsilon} = 8\pi G\, \epsilon^{ab}\boldsymbol{\varepsilon}_{\epsilon\alpha\beta\rho}\left(\partial_\gamma \overline{T}^{\rho}_{\ b\,\delta} - \partial_\delta \overline{T}^{\rho}_{\ b\,\gamma}\right)$$
$$\iff \partial_\mu R_a^{\gamma\delta\rho\mu} = 8\pi G\left(\partial^\delta \overline{T}_a^{\rho\gamma} - \partial^\gamma \overline{T}_a^{\rho\delta}\right), \tag{B.10}$$

while the contracted Bianchi identities are

$$\partial_\nu G_a^{\mu\nu} = 0. \tag{B.11}$$

Let

$$K^{\lambda\tau}{}_{\mu\nu\rho\sigma}[R^a_{\lambda\tau}] = \frac{1}{2}\left[\eta_{\mu\rho}R^a_{\nu\sigma} + \eta_{\nu\sigma}R^a_{\mu\rho} - \eta_{\mu\sigma}R^a_{\nu\rho} - \eta_{\nu\rho}R^a_{\mu\sigma}\right] -$$
$$- \frac{R^a}{6}\left[\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right]. \tag{B.12}$$

Defining

$$\widetilde{R}^a_{\mu\nu\rho\sigma} = R^a_{\mu\nu\rho\sigma} - \frac{1}{2}\epsilon^{ab}\boldsymbol{\varepsilon}_{\rho\sigma\alpha\beta}K^{\lambda\tau}{}_{\mu\nu}{}^{\alpha\beta}[R_{b\lambda\tau}], \tag{B.13}$$

the tensor $\widetilde{R}^a_{\mu\nu\rho\sigma}$ is skew in the first and last pairs of indices, satisfies the cyclic identity because $\boldsymbol{\varepsilon}^{\gamma\nu\rho\sigma}R^a_{\mu\nu\rho\sigma} = \boldsymbol{\varepsilon}^{\gamma\nu\rho\sigma}\frac{1}{2}\epsilon^{ab}\boldsymbol{\varepsilon}_{\rho\sigma\alpha\beta}K^{\lambda\tau}{}_{\mu\nu}{}^{\alpha\beta}[R_{b\lambda\tau}]$ and, as a consequence, is also symmetric in the exchange of the first and last pair of indices, $\widetilde{R}^a_{\mu\nu\rho\sigma} = \widetilde{R}^a_{\rho\sigma\mu\nu}$. The associated Ricci tensors $\widetilde{R}^a_{\nu\sigma} = R^a_{\nu\sigma} - \frac{1}{2}\epsilon^{ab}\boldsymbol{\varepsilon}_{\nu\sigma\mu\alpha}R_b^{\mu\alpha}$ is then symmetric, $\widetilde{R}^a_{\nu\sigma} = \widetilde{R}^a_{\sigma\nu}$. It follows that $\widetilde{R}^a_{\nu\sigma} = R^a_{(\nu\sigma)}$ and $R^a_{[\nu\sigma]} = \frac{1}{2}\epsilon^{ab}\boldsymbol{\varepsilon}_{\nu\sigma\mu\alpha}R_b^{\mu\alpha}$. The Weyl tensors are then defined as usual in terms of $\widetilde{R}^a_{\mu\nu\rho\sigma}$,

$$C^a_{\mu\nu\rho\sigma} = \widetilde{R}^a_{\mu\nu\rho\sigma} - K^{\lambda\tau}{}_{\mu\nu\rho\sigma}[\widetilde{R}^a_{\lambda\tau}], \tag{B.14}$$

and satisfy all standard symmetry properties: skew-symmetry in the first and last pairs of indices, tracelessness (because $\widetilde{R}^a_{\nu\sigma} = K^{\lambda\tau\,\mu}{}_{\nu\mu\sigma}[\widetilde{R}^a_{\lambda\tau}]$), the cyclic identity (because $\epsilon^{\gamma\nu\rho\sigma}K^{\lambda\tau}{}_{\mu\nu\rho\sigma}[\widetilde{R}^a_{\lambda\tau}] = 0$), which implies also symmetry in the exchange of the first and last pair of indices,

$$C^a_{\mu\nu\rho\sigma} = -C^a_{\nu\mu\rho\sigma} = -C^a_{\mu\nu\sigma\rho}, \tag{B.15}$$
$$C^{\mu a}_{\nu\mu\sigma} = 0, \quad \epsilon^{\gamma\nu\rho\sigma}C^a_{\mu\nu\rho\sigma} = 0, \quad C^a_{\mu\nu\rho\sigma} = C^a_{\rho\sigma\mu\nu}. \tag{B.16}$$

As usual, the 10 independent components of the Weyl tensor can be parametrized by the electric and magnetic components $E^a_{mn} \equiv (E_{mn}, B_{mn})$, symmetric and traceless tensors defined by

$$E^a_{mn} = C^a_{0m0n} = \frac{1}{2}\epsilon_{njk}\epsilon^{ab}C_{b\,0m}{}^{jk}. \tag{B.17}$$

Putting all definitions together, the relation between the Riemann and Weyl tensors is

$$R^a_{\mu\nu\rho\sigma} = C^a_{\mu\nu\rho\sigma} + K^{\lambda\tau}{}_{\mu\nu\rho\sigma}[R^a_{\lambda\tau}] + \frac{1}{2}\epsilon^{ab}\boldsymbol{\varepsilon}_{\rho\sigma\alpha\beta}K^{\lambda\tau}{}_{\mu\nu}{}^{\alpha\beta}[R_{b(\lambda\tau)}] \tag{B.18}$$

$$= C^a_{\mu\nu\rho\sigma} + K^{\lambda\tau}{}_{\mu\nu\rho\sigma}[R^a_{(\lambda\tau)}] + \frac{1}{2}\epsilon^{ab}\boldsymbol{\varepsilon}_{\rho\sigma\alpha\beta}K^{\lambda\tau}{}_{\mu\nu}{}^{\alpha\beta}[R_{b\lambda\tau}]. \tag{B.19}$$

In particular, it follows that the 36 independent components of the Riemman tensor $R^1_{\mu\nu\rho\sigma}$ can be parameterized by the 10 independent components of the Weyl tensor $C^1_{\mu\nu\rho\sigma}$, the 16 components of the Ricci tensor $R^1_{\lambda\tau}$, and the 10 components of $R^2_{(\lambda\tau)}$.

If we define

$$\mathcal{E}^a_{mn} = R^a_{0(m|0|n)}, \quad \mathcal{F}^{am} = \frac{1}{2}\epsilon^{mjk}R^a_{0[j|0|k]}, \quad \mathcal{R}^a_{mn} = R^a_{(mn)} + \mathcal{E}^a_{mn} \tag{B.20}$$

the parameterization consisting in choosing the symmetric tensors $\mathcal{E}^a_{mn}, \mathcal{R}^a_{mn}$ (24 components), $\mathcal{F}^a_m$, (6 components), and $R^1_{[\mu\nu]}(= {}^*R^2_{[\mu\nu]})$ (6 components) is more useful for our purpose. That all tensors can be reconstructed from these variables follows from the fact that

$$R^a_{0m} = -2\epsilon^{ab}\mathcal{F}_{bm}, \quad R^a_{00} = \mathcal{E}^a, \quad R^a_{(mn)} = \mathcal{R}^a_{mn} - \mathcal{E}^a_{mn}. \tag{B.21}$$

This means that the symmetric part of the Ricci tensors can be reconstructed from the variables. Since the antisymmetric parts belong to the variables, so can the complete Ricci tensors $R^a_{\mu\nu}$. Using now (B.18) and definitions (B.17), (B.20), (B.12), we find

$$E^a_{mn} = \frac{1}{2}(\mathcal{E}^a_{mn} + \mathcal{R}^a_{mn}) - \frac{\delta_{mn}}{6}(\mathcal{E}^a + \mathcal{R}^a). \tag{B.22}$$

It follows that the Weyl tensors and then, using again (B.18), the Riemann tensors can be reconstructed.

In terms of the new parameterization, the equations of motion (B.9) read $R^a_{[\mu\nu]} = 0$ and

$$-2\epsilon^{ab}\mathcal{F}_{bm} = 8\pi G T^a_{0m}, \tag{B.23}$$

$$\frac{1}{2}\mathcal{R}^a = 8\pi G T^a_{00}, \tag{B.24}$$

$$\mathcal{R}^a_{mn} - \mathcal{E}^a_{mn} + \delta_{mn}(\mathcal{E}^a - \frac{1}{2}\mathcal{R}^a) = 8\pi G T^a_{mn}. \tag{B.25}$$

Using these equations of motion, the Bianchi identities (B.10) are equivalent to

$$\partial^k(\epsilon_{ikm}\mathcal{F}^{am} + \mathcal{R}^a_{ik}) = \frac{1}{2}\partial_i\mathcal{R}^a, \tag{B.26}$$

$$2\epsilon^{ab}\partial_0\mathcal{F}_{bm} = \partial^n(\mathcal{E}^a_{mn} + \epsilon_{mnk}\mathcal{F}^{ak}) - \partial_m\mathcal{E}^a, \tag{B.27}$$

$$\begin{aligned}\partial_0\mathcal{R}^a_{ik} &= \frac{1}{2}\epsilon^{ab}\big[\epsilon_{kjl}\partial^j\mathcal{E}_{bi}{}^l + \epsilon_{ijl}\partial^j\mathcal{E}_{bk}{}^l - 2\delta_{ik}\partial_j\mathcal{F}^j_b - \partial_i\mathcal{F}_{bk} - \partial_k\mathcal{F}_{bi}\big] \iff \\ \epsilon^{ab}\partial_0(\mathcal{R}^{ik}_b - \frac{1}{2}\delta^{ik}\mathcal{R}_b) &= -\frac{1}{2}\big[\epsilon^{klm}\partial_l\mathcal{E}^{a\,i}_m + \epsilon^{ilm}\partial_l\mathcal{E}^{a\,k}_m + 2\delta^{ik}\partial^j\mathcal{F}^a_j - \partial^i\mathcal{F}^{ak} - \partial^k\mathcal{F}^{ai}\big].\end{aligned} \tag{B.28}$$

B.2 Canonical expressions

We will now express the Riemann tensor in terms of the canonical variables in such a way that the covariant equations (B.23)-(B.28) coincide with the Hamiltonian equations deriving from (4.2).

From the constraints with sources, we find

$$\mathcal{R}^a = \partial^m\partial^n h^a_{mn} - \Delta h^a = -\Delta^2 C^a, \tag{B.29}$$

$$\mathcal{F}^a_m = \frac{1}{2}\Delta\partial^n H^a_{mn}. \tag{B.30}$$

Assuming Δ to be invertible, which we do in the rest of this appendix, $\mathcal{R}^a$ and C^a, respectively $\mathcal{F}^a_m$ and $\partial^n H^a_{mn}$ determine each other. By taking the divergence, the Bianchi identity (B.26) implies that

$$\partial^m\partial^n\mathcal{R}^a_{mn} = -\frac{1}{2}\Delta^3 C^a.$$

Similarily, the Bianchi identity (B.27) implies in particular that $\Delta\mathcal{E}^a - \partial^m\partial^n\mathcal{E}^a_{mn} = \epsilon^{ab}\partial_0\Delta\partial^m\partial^n H_{bmn}$. When combined with (B.25), the equations of motion following from variation with respect to C^a read

$$\frac{1}{2}\Delta^3 C^a + \epsilon^{ab}\Delta\partial_0(\Delta H_b - \partial^m\partial^n H_{bmn}) + 2\Delta^2 n^a = \Delta\mathcal{E}^a - \partial^m\partial^n(\mathcal{R}^a_{mn} - \mathcal{E}^a_{mn}).$$

When combined with the previous relations, they imply that

$$\mathcal{E}^a = -\frac{1}{2}\epsilon^{ab}\partial_0\Delta H_b + \Delta n^a,$$

$$\partial^m\partial^n\mathcal{E}^a_{mn} = -\frac{1}{2}\epsilon^{ab}\partial_0\Delta(\Delta H_b - 2\partial^m\partial^n H_{bmn}) + \Delta^2 n^a.$$

The rest of the Bianchi identities (B.26), (B.27) are taken into account by applying a curl. This gives $\epsilon^{rsi}\partial_s\partial^k\mathcal{R}^a_{ik} = \frac{1}{2}\Delta(\Delta\partial^k H^{ar}_k - \partial^r\partial^m\partial^n H^a_{mn})$ and $\epsilon^{rsi}\partial_s\partial^k\mathcal{E}^a_{ik} = \epsilon^{rsi}2\epsilon^{ab}\partial_0\partial_s\mathcal{F}_{bi} - \partial^r\partial^k\mathcal{F}^a_k + \Delta\mathcal{F}^{ar}$. Yet another curl gives $\partial_k\partial^m\partial^n\mathcal{R}^a_{mn} - \Delta\partial^n\mathcal{R}^a_{kn} = \frac{1}{2}\epsilon_{klr}\partial^l\Delta^2\partial^n H^{ar}_n$ and $\partial_k\partial^m\partial^n\mathcal{E}^a_{mn} - \Delta\partial^n\mathcal{E}^a_{kn} = 2\epsilon^{ab}\partial_0(\partial_k\partial^n\mathcal{F}_{bn} - \Delta\mathcal{F}_{bk}) + \epsilon_{klr}\partial^l\Delta\mathcal{F}^{ar}$. Using the previous relations we then get

$$\partial^n\mathcal{R}^a_{kn} = -\frac{1}{2}\partial_k\Delta^2 C^a - \frac{1}{2}\epsilon_{klr}\partial^l\Delta\partial^n H^{ar}_n,$$

$$\partial^n\mathcal{E}^a_{kn} = \epsilon^{ab}\partial_0\Delta(-\frac{1}{2}\partial_k H_b + \partial^n H_{bkn}) + \partial_k\Delta n^a - \frac{1}{2}\epsilon_{klr}\partial^l\Delta\partial^n H^{ar}_n.$$

The equations of motion following from variation with respect to A^a_m are then identically satisfied.

Defining $\mathcal{D}^a_{mn} = \mathcal{R}^a_{mn} - \mathcal{E}^a_{mn}$ and using definition (2.12) of $\mathcal{P}^{TT}$ combined with (B.25), the equations of motion following from variation with respect to H^a_{mn} read

$$\epsilon_{ab}\partial_0\Big[2\left(\mathcal{P}^{TT}H^b\right)_{mn} + \partial_m\Delta A^b_n + \partial_n\Delta A^b_m + \frac{1}{2}(\delta_{mn}\Delta - \partial_m\partial_n)C^b\Big] - \epsilon_{ab}\Delta(\partial_m n^b_n + \partial_n n^b_m) -$$
$$- 2\Delta^2 H^a_{mn} + \delta_{mn}\Delta^2 H^a = -\epsilon_{mpq}\partial^p\mathcal{D}^q_{an} - \epsilon_{npq}\partial^p\mathcal{D}^q_{am}. \tag{B.31}$$

Taking into account definition (2.12) and previous relations, we can extract

$$-\Delta^{-1}\left(\mathcal{P}^{TT}\mathcal{D}^a\right)_{mn}=\frac{1}{2}\epsilon_{ab}\partial_0\Big[2\left(\mathcal{P}^{TT}H^b\right)_{mn}-\epsilon_{mpq}\partial_n\partial^p\partial^r H^{bq}_r-\epsilon_{npq}\partial_m\partial^p\partial^r H^{bq}_r+\\+\partial_m\Delta A^b_n+\partial_n\Delta A^b_m+\frac{1}{2}(\delta_{mn}\Delta-\partial_m\partial_n)C^b\Big]-\epsilon_{ab}\Delta(\partial_m n^b_n+\partial_n n^b_m)-\\-\Delta^2 H^a_{mn}+\frac{1}{2}\delta_{mn}\Delta^2 H^a. \quad \text{(B.32)}$$

In order to extract the remaining information from (B.31), we first apply $\delta^{mn}\Delta-\partial^m\partial^n$ to get

$$\epsilon_{ab}\partial_0\Delta C^b+2\partial^m\partial^n H_{amn}=0, \quad \text{(B.33)}$$

and then a divergence ∂^m giving

$$\epsilon_{ab}\partial_0(\Delta A^b_n-\epsilon_{npq}\partial^p\partial^k H^{bq}_k)=\epsilon_{ab}\Delta n^b_n+2\Delta\partial^k H^k_{an}-\frac{1}{2}\partial_n\Delta H^a-\partial_n\partial^k\partial^l H^a_{kl}. \quad \text{(B.34)}$$

We can now inject the latter relations into (B.31) and use (2.14), (2.6) to get

$$\mathcal{D}^{aTT}_{mn}=-\epsilon^{ab}\partial_0\Delta H^{TT}_{bmn}-\left(\mathcal{P}^{TT}H^a\right)_{mn}, \quad \text{(B.35)}$$

$$\mathcal{D}^a_{mn}=-\epsilon^{ab}\partial_0\Delta\Big[H_{bmn}-\frac{1}{2}\delta_{mn}H_b\Big]-\left(\mathcal{P}^{TT}H^a\right)_{mn}-\partial_m\partial_n n^a-\\-\frac{1}{4}(\delta_{mn}\Delta+\partial_m\partial_n)\Delta C^a. \quad \text{(B.36)}$$

Injecting into the second form of the last Bianchi identity (B.28) and using previous relations gives

$$\epsilon_{ab}\partial_0\mathcal{R}^b_{ij}=-\left(\mathcal{O}\mathcal{R}_a\right)_{ij}+\Delta^2 H^{TT}_{aij}+\frac{1}{4}\Delta\partial_i\partial^k H_{akj}+\frac{1}{4}\Delta\partial_j\partial^k H_{aki}-\frac{1}{2}\partial_i\partial_j\partial^k\partial^l H_{akl}\\-\frac{1}{2}\epsilon_{ab}\partial_0\Big[\epsilon_{iqn}\partial^q\Delta H^{bn}_j+\epsilon_{jqn}\partial^q\Delta H^{bn}_i+\frac{1}{2}(\delta_{ij}\Delta+\partial_i\partial_j)\Delta C^b\Big]. \quad \text{(B.37)}$$

Identifying the terms with time derivatives gives

$$\mathcal{R}^a_{ij}=-\frac{1}{2}\Big[\epsilon_{iqn}\partial^q\Delta H^{an}_j+\epsilon_{jqn}\partial^q\Delta H^{an}_i+\frac{1}{2}(\delta_{ij}\Delta+\partial_i\partial_j)\Delta C^a\Big]\\=\frac{1}{2}\Big[\partial_i\partial^k h^a_{kj}+\partial_j\partial^k h^a_{ki}-\partial_i\partial_j h^a-\Delta h^a_{ij}-\epsilon_{ikl}\partial^k\partial^p\partial_j H^{al}_p-\epsilon_{jkl}\partial^k\partial^p\partial_i H^{al}_p\Big]. \quad \text{(B.38)}$$

The terms without time derivatives in (B.37) then cancel identically. Together with (B.36) this then finally gives

$$\mathcal{E}^{ij}_a=\epsilon_{ab}\partial_0\Delta\Big[H^{bij}-\frac{1}{2}\delta^{ij}H^b\Big]+\partial^i\partial^j n_a-\frac{1}{2}\epsilon^{ikl}\partial_k\partial^j\partial^p H_{alp}-\frac{1}{2}\epsilon^{jkl}\partial_k\partial^i\partial^p H_{alp}\\=-\epsilon_{ab}\partial_0(\pi^{bij}-\frac{1}{2}\delta^{ij}\pi^b)+\partial^i\partial^j n_a-\frac{1}{2}\epsilon^{ikl}\partial_k\partial^j\partial^p H_{alp}-\frac{1}{2}\epsilon^{jkl}\partial_k\partial^i\partial^p H_{alp}. \quad \text{(B.39)}$$

B.3 Riemann tensor for linearized Taub-NUT

Following for instance [39] section $A_1.2$ and using a regularization in Fourier space, we find for the gravitational dyon at rest at the origin discussed in section **4.2**,

$$\mathcal{R}^a_{ij} = GM^a\big[\frac{16\pi}{3}\delta_{ij}\delta^3(x) + \frac{\eta(r)}{r^3}(\delta_{ij} - \frac{3x_ix_j}{r^2})\big], \tag{B.40}$$

$$\mathcal{E}^a_{ij} = GM^a\big[\frac{4\pi}{3}\delta_{ij}\delta^3(x) + \frac{\eta(r)}{r^3}(\delta_{ij} - \frac{3x_ix_j}{r^2})\big], \tag{B.41}$$

where $\eta(r)$ is a regularizing function that suppresses the divergence at the origin and is 1 away from the origin. We then find

$$R^a_{00} = GM^a 4\pi\delta^3(x), \quad R^a_{ij} = GM^a 4\pi\delta_{ij}\delta^3(x), \tag{B.42}$$

$$E^a_{ij} = GM^a\frac{\eta(x)}{r^3}(\delta_{ij} - \frac{3x_ix_j}{r^2}), \tag{B.43}$$

and all other components of $R^a_{\mu\nu}$ vanishing. For the Riemann tensor, this implies

$$R^a_{0i0j} = GM^a\big[\frac{4\pi}{3}\delta_{ij}\delta^3(x) + \frac{\eta(x)}{r^3}(\delta_{ij} - \frac{3x_ix_j}{r^2})\big], \tag{B.44}$$

$$R^a_{0ijk} = -\epsilon^{ab}\epsilon_{jk}{}^{l}GM^a\big[\frac{4\pi}{3}\delta_{il}\delta^3(x) + \frac{\eta(x)}{r^3}(\delta_{il} - \frac{3x_ix_l}{r^2})\big], \tag{B.45}$$

with all other components obtained through the on-shell symmetries of the Riemann tensor. This is the usual Riemann tensor for the linearized Taub-NUT solution.

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